Math. Subj. Class. (2000): 03B25, 11U05

Division-ample sets and the Diophantine problem for rings of integers

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Abstract. We prove that Hilbert's Tenth Problem for a ring of integers in a number field K has a negative answer if K satisfies two arithmetical conditions (existence of a so-called *division-ample* set of integers and of an elliptic curve of rank one over K). We relate division-ample sets to arithmetic of abelian varieties.

Introduction.

Let K be a number field and let $\emptyset_K$ be its ring of integers. *Hilbert's Tenth Problem* or *the diophantine problem* for $\emptyset_K$ is the following: is there an algorithm (on a Turing machine) that decides whether an arbitary diophantine equation with coefficients in $\emptyset_K$ has a solution in $\emptyset_K$.

The answer to this problem is known to be negative if $K = \mathbf{Q}$ ([5]), and for several other such K (such as imaginary quadratic number fields [6], totally real fields [7], abelian number fields [11]) by reduction to the case $K = \mathbf{Q}$. This reduction consists in finding a *diophantine model* (cf. [2]) for integer arithmetic over $\emptyset_K$. The problem is open for general number fields (for a survey see [9] and [12]), but has been solved conditionally, e.g. by Poonen [10] (who shows that the set if rational integers is diophantine over $\emptyset_K$ if there exists an elliptic curve over $\mathbf{Q}$ that has rank one over both $\mathbf{Q}$ and K). In this paper, we give a more general condition as follows:

Theorem. *The diophantine problem for the ring of integers $\emptyset_K$ of a number field K has a negative answer if the following exist:*

(i) an elliptic curve defined over K of rank one over K;

(ii) a division-ample *set $A \subseteq \emptyset_K$.*

A set $A \subseteq \mathcal{O}_K$ is called *division-ample* if the following three conditions are satisfied:

$\diamond$ (diophantineness) A is a diophantine subset of $\mathcal{O}_K$;

$\diamond$ (divisibility-density) Any $x \in \mathcal{O}_K$ divides an element of A;

$\diamond$ (norm-boundedness) There exists an integer $\ell > 0$, such that for any $a \in A$, there is an integer $\widetilde{a} \in \mathbf{Z}$ with $\widetilde{a}$ dividing a and $|N(a)| \leq |\widetilde{a}|^\ell$.

Proposition. *A division ample set exists if either*

(i) there exists an abelian variety G over $\mathbf{Q}$ such that

$$\operatorname{rk} G(\mathbf{Q}) = \operatorname{rk} G(K) > 0; \text{ or}$$

(ii) there exists a commutative (not necessarily complete) group variety G over $\mathbf{Z}$ such that $G(\mathcal{O}_K)$ is finitely generated and such that $\operatorname{rk} G(\mathbf{Q}) = \operatorname{rk} G(K) > 0$.

From (i) in this proposition, it follows that our theorem includes that of Poonen, but it isolates the notion of "division-ampleness" and shows it can be satisfied in a broader context. It would for example be interesting to construct, for a given number field K, a curve over $\mathbf{Q}$ such that it's Jacobian satisfies this condition.

As we will prove below, part (ii) of this proposition is satisfied for the relative norm one torus $G = \ker(N_K^{KL})$ for a number field L linearly disjoint from K, if K is quadratic imaginary (choosing L totally real).

It would be interesting to know other division-ample sets, in particular, such that are not subsets of the integers.

The proof of the main theorem will use divisibility on elliptic curves and a lemma from algebraic number theory of Denef and Lipshitz. Some of our arguments are similar to ones in [10], but we have avoided continuous reference both for reasons of completeness and because our results have been obtained independently.

1. Lemmas on number fields

In this section we collect a few facts about general number fields which will play a rôle in subsequent proofs. Fix K to be a number field, let $\mathcal{O} = \mathcal{O}_K$ be its ring of integers, and let h denote the class number of $\mathcal{O}$. Let $N = N_{\mathbf{Q}}^K$ be the norm from K to $\mathbf{Q}$, and let $n = [K : \mathbf{Q}]$ denote the degree of K. Let $|$ denote "divides" in $\mathcal{O}$.

First of all, we will say a subset $S \subseteq K^n$ is "diophantine over $\mathcal{O}$" if its set of representatives $\widetilde{S} \subseteq (\mathcal{O} \times (\mathcal{O} - \{0\}))^n$ given by

$$\widetilde{S} := \{(a_i, b_i)_{i=1}^n \in (\mathcal{O} \times (\mathcal{O} - \{0\}))^n \mid (a_i/b_i)_{i=1}^n \in S\}$$

is diophantine over $\mathcal{O}$. Recall that "$x \neq 0$" is diophantine over $\mathcal{O}$ ([8] Prop. 1(b)), hence S is diophantine over $\mathcal{O}$ if and only if it is diophantine over K.

Recall that there is no unique factorisation in general number fields, but we can use the following valuation-theoretic remedy:

1.1 Definition. Let $x \in K$. If $x^h = \frac{a}{b}$ for $a, b \in \O$ with $(a, b) = 1$ (the ideal generated by a and b), we say that $a = \mathrm{wn}(x)$ is *a weak numerator* and $b = \mathrm{wd}(x)$ is *a weak denominator* for x.

1.2 Lemma. *(i) For any $x \in K$ a weak numerator and a weak denominator exists and is unique up to units.*

(ii) for any valuation, $v(x) > 0 \iff v(\mathrm{wn}(x)) > 0$ [and then $v(\mathrm{wn}(x)) = hv(x)$], and $v(x) < 0 \iff v(\mathrm{wd}(x)) > 0$ [and then $v(\mathrm{wd}(x)) = -hv(x)$].

(iii) For $a \in \O, x \in K$, "$a = \mathrm{wn}(x)$" and "$a = \mathrm{wd}(x)$" are diophantine over $\O$.

Proof. Since $\O$ is a Dedekind ring, (x) has a unique factorisation in fractional ideals

$$(x) = \mathfrak{p}_1 \cdots \mathfrak{p}_r \cdot \mathfrak{q}_1^{-1} \cdots \mathfrak{q}_s^{-1}.$$

We let a be a generator for the principal ideal $(\mathfrak{p}_1 \cdots \mathfrak{p}_r)^h$ and b a generator for $(\mathfrak{q}_1 \cdots \mathfrak{q}_s)^h$; these are obviously weak numerator/denominator for x. Uniqueness, (ii) and (iii) are obvious. □

1.3 Lemma. (Denef-Lipshitz [8])

(i) If $u \in \mathbf{Z} - \{0\}$ and $\xi \in \O$ satisfy the divisibility condition

$$2^{n!+1} \prod_{i=0}^{n!-1} (\xi + i)^{n!} \mid u$$

then for any embedding $\sigma : K \hookrightarrow \mathbf{C}$

$$(*)_u \qquad |\sigma(\xi)| \leq \frac{1}{2} \sqrt[n!]{|N(u)|}.$$

(ii) If $\widetilde{u} \in \mathbf{Z} - \{0\}, q \in \mathbf{Z}$ and $\xi \in \O$ satisfy $(*)_{\widetilde{u}}$ for any embedding $\sigma : K \hookrightarrow \mathbf{C}$ and $\xi \equiv q \bmod \widetilde{u}$, then $\xi \in \mathbf{Z}$.

Proof. Easy to extract from the proof of Lemma 1 in [8]. □

2. Lemmas on elliptic curves

Let E denote an elliptic curve of rank one over K, written in Weierstrass form as

$$E \ : \ y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

let T be the order of the torsion group of $E(K)$, and let P be a generator for the free part of $E(K)$. Define $x_n, y_n \in K$ by $nP = (x_n, y_n)$.

2.1 Lemma. *For any integer r the set $rE(K)$ is diophantine over K and, if r is divisible by T, then $rE(K) = \langle rP \rangle \cong (\mathbf{Z}, +)$.*

Proof. A point $(x, y) \in K \times K$ belongs to $rE(K) - \{0\}$ if and only if $\exists\ (x_0, y_0) \in E(K) \ : \ (x, y) = r(x_0, y_0)$. As the addition formulæ on E are

algebraic with coefficients from K, this is a diophantine relation. The last statement is obvious. $\square$

2.2 Lemma. *There exists an integer $r > 0$ such that for any non-zero integers $m, n \in \mathbf{Z}$, m divides n if and only if* $\mathrm{wd}(x_{rm}) | \mathrm{wd}(x_{rn})$.

Proof. We reduce the claim to a statement about valuations using lemma 1.2(ii). The theory of the formal group associated to E implies that if $n = mt$ and v is a finite valuation of K such that $v(x_{rm}) < 0$, then $v(x_{rmt}) = v(x_{rm}) - 2v(t) \leq v(x_{rm})$ ([13] VII.2.2).

For the converse, we start by choosing r_0 in such a way that $r_0 P$ is non-singular modulo all valuations v on K. By the theorem of Kodaira-Néron ([13], VII.6.1), such r_0 exists and it actually suffices to take $r_0 = 4\prod_v v(\Delta_E)$, where Δ_E is the minimal discriminant of E, and the product runs over all finite valuations on K for which $v(\Delta_E) \neq 0$. Note that then, $v(x_{r_0 n}) < 0 \iff r_0 n P = 0$ in the group E_v of non-singular points of E modulo v.

We claim that for an arbitrary finite valuation v on K, if $v(x_{r_0 n}) < 0$ and $v(x_{r_0 m}) < 0$, then $v(x_{(r_0 m, r_0 n)}) < 0$, where $(\cdot, \cdot)$ denotes the gcd in $\mathbf{Z}$. Indeed, the hypothesis means $r_0 m P = r_0 n P = 0$ in E_v. Since there exist integers $a, b \in \mathbf{Z}$ with $(r_0 m, r_0 n) = a r_0 m + b r_0 n$, we find $(r_0 m, r_0 n) P = 0$ in E_v, and hence the claim.

The main theorem of [1] states that for any sufficiently large $M (\geq M_0)$, there exists a finite valuation v such that $v(x_M) < 0$ but $v(x_i) \geq 0$ for all $i < M$. We choose $r = r_0 M_0$. Pick such a valuation v for $M = rm$. The hypothesis implies that $v(x_{rn}) < 0$ and hence $v(x_{r(m,n)}) < 0$. But $r(m, n) \leq rm$ and $v(x_i) \geq 0$ for any $i < rm$. Hence $r(m, n) = rm$ so m divides n. $\square$

2.3 Lemma. *Any $\xi \in \text{Ø} - \{0\}$ divides the weak denominator of some x_n.*

Proof. Let v be a valuation lying over the prime p_v of $\mathbf{Z}$, and assume that $v(\xi) = e_v > 0$. Then the group of non-singular points on E modulo v is finite, hence there exists an n_v such that $n_v r P = 0$ in this group, i.e., $v(x_{n_v r}) < 0$. By the formal group law formula, we see that $v(x_{p_v^{e_v} n_v r}) = v(x_{n_v r}) - 2v(p_v^{e_v}) < -e_v$. Putting

$$n = r \cdot \prod_{v(\xi) > 0} n_v p_v^{e_v}$$

will then certainly suffice. $\square$

2.4 Lemma. *Let m, n, q be integers with $n = mq$. Then*

$$\mathrm{wd}(x_m) | \mathrm{wn}(\frac{x_n y_m}{y_n x_m} - q).$$

Proof. The formal power series expansion for addition on E around 0 ([13], IV.2.3) implies that $\frac{x_n}{y_n} = q\frac{x_m}{y_m} + O((\frac{x_m}{y_m})^2)$, from which the result follows. ☐

3. Proof of the main theorem

Let $\xi \in \emptyset$. Given an elliptic curve E of rank one over K as in the main theorem, we use the notation from section 2 for this E — in particular, choose a suitable r such that lemma 2.2 applies; we also choose ℓ which comes with the definition of A. We claim that the following formulæ give a diophantine definition of $\mathbf{Z}$ in $\emptyset$:

$$\xi \in \mathbf{Z} \iff \exists\, m, n \in rT\mathbf{Z},\ \exists\, u \in A - \{0\} \begin{cases} (1) & m|n \\ (2) & 2^{n!+1} \prod_{i=0}^{n!-1} (\xi^{\ell n!} + i)^{n!} | u \\ (3) & u^h |\, \mathrm{wd}(x_m) \\ (4) & \mathrm{wd}(x_m) |\, \mathrm{wn}(\frac{x_n y_m}{y_m x_n} - \xi) \end{cases}$$

3.1 Any $\xi \in \mathbf{Z}$ satisfies the relations. If $\xi \in \mathbf{Z}$, then a u satisfying (2) exists because A is division-dense. By lemma 2.3, there exists an m satisfying (3) for this u. Define $n = m\xi$ for this m. Then (1) is automatic and (4) is the contents of lemma 2.4.

3.2 A ξ satisfying the relations is integral. Let $q \in \mathbf{Z}$ satisfy $n = qm$ (which exists by (1)). Then lemma 2.4 implies that

$$\mathrm{wd}(x_m) |\, \mathrm{wn}(\frac{x_n y_m}{x_m y_n} - q),$$

which can be combined with (4) using the non-archimedean triangle inequality to give

$$\mathrm{wd}(x_m) |\, \mathrm{wn}(\xi - q) = (\xi - q)^h.$$

By (3), then also $u|\xi - q$.

By norm-boundedness of A we can find $\widetilde{u} \in \mathbf{Z}$ such that $\widetilde{u}|u$ and $|N(u)| \leq \widetilde{u}^{\ell}$. We still have

$$(*)\quad \xi \equiv q \bmod \widetilde{u};\quad \widetilde{u}, q \in \mathbf{Z}.$$

Condition (2) implies that Lemma 1.3(i) can be applied with $\xi^{\ell n!}$ in place of ξ, so for any complex embedding σ of K we find

$$(**)\quad |\sigma(\xi)| \leq \frac{1}{2}|N(u)|^{\frac{1}{\ell n!}} \leq \frac{1}{2}N(\widetilde{u})^{\frac{1}{n!}}.$$

Because of $(*)$ and $(**)$, we can apply Lemma 1.3(ii) to conclude $\xi \in \mathbf{Z}$.

3.3 The relations (1)-(4) are diophantine over $\emptyset$. By 1.2 and 2.1, for $a \in \emptyset$, the relations $\exists n \in rT\mathbf{Z} : a = \mathrm{wn}(x_n)$ and $\exists n \in rT\mathbf{Z}$:

$a = \mathrm{wd}(x_n)$ are diophantine. By the diophantineness of A, the membership $u \in A$ is diophantine, and $u \neq 0$ is diophantine ([8], Prop. 1(b)). Condition (1) is diophantine because of Lemma 2.2. Conditions (2)-(4) are obviously diophantine using 1.2. $\square$

4. Proof of the proposition and discussion of division-ample sets

4.1 Rank-preservation over Q. We use [3] as a general reference on abelian varieties and formal groups. Suppose there exists an abelian variety G of dimension d over $\mathbf{Q}$ such that $\mathrm{rk}\, G(\mathbf{Q}) = \mathrm{rk}\, G(K) > 0$ (note that $G(K)$ is finitely generated by the Mordell-Weil theorem). Let T denote the (finite) order of the torsion of $G(K)$ and consider the free group $TG(K) \cong \mathbf{Z}^r$. The assumption implies that $G(\mathbf{Q})$ is of finite index $[G(K) : G(\mathbf{Q})]$ in $G(K)$. The choice of an ample line bundle on G gives rise to a projective embedding of G in some projective space with coordinates $\langle x_i \rangle_{i=1}^N$, where G is cut out by finitely many polynomial equations and the addition on G is algebraic in those coordinates. Suppose $\{t_i\}$ are algebraic function of the coordinates, and local uniformizers at the unit $\mathbf{0} = (1 : 0 : \ldots : 0)$ of G (i.e., $\hat{\mathcal{O}}_{G,\mathbf{0}} = \mathbf{Q}[[t_1, \ldots, t_d]]$), and define

$$A_G := \{\mathrm{wd}(\prod_{i=2}^N t_i(P)) \ : \ P \in T[G(K) : G(\mathbf{Q})] \cdot G(K) \text{ and } t_1(P) = 1\}.$$

We claim that A_G is division-ample. Indeed, the three conditions are satisfied:

(a) A_G is obviously diophantine over $\mathcal{O}$ (the diophantine definition comes from the chosen embedding of G).

(b) The analogue of lemma 2.3 remains valid:

Claim. *A_G is divisibility-dense.*

Proof. Since any $\xi \in \mathcal{O} - \{0\}$ divides its norm, it suffice to prove that any integer $z \in \mathbf{Z} - \{0\}$ divides an element of A_G. Given a minimal model $\mathcal{G}_p$ for G over a p-adic field $\mathbf{Q}_p$, let $\mathcal{G}_{p,0}$ denote the group of points whose reduction is non-singular modulo p. Then $\mathcal{G}_{p,0}$ is a clopen subset of $\mathcal{G}_p$, so $\mathcal{G}_p/\mathcal{G}_{p,0}$ is finite (and non-trivial only for the finite set of primes for which $\mathcal{G}_p$ has bad reduction). Hence we can choose a finite r so large that rP_i is non-singular modulo all primes for all generators P_i of $G(\mathbf{Q})$. Pick a prime $p|z$, then since the group of non-singular points on G modulo p is finite, there exists n_p such that $n_p r P = 0$ in this group, i.e., $v_p(t_i(P)) > 0$ for all i. The formal group $\hat{G}_{\mathbf{0}}$ of G around $\mathbf{0}$ (defined by the power series that give the addition in terms of $\{t_i\}$) is a formal torus in characteristic zero, and

hence admits for any $N > 0$ a formal logarithmic isomorphism to a power of the additive group

$$\phi : \hat{G}_{\mathbf{0}}(p^N \mathbf{Z}_p) \cong \hat{\mathbf{G}}_a^d(p^N \mathbf{Z}_p)$$

preserving valuations. Hence for any n,

$$\begin{aligned} v(t_i(nP)) &= v(\phi(t_i(nP))) = v(n(\phi(t_i(P))) = v(n) + v(\phi(t_i(P)) \\ &= v(n) + v(t_i(P)), \end{aligned}$$

and we can find n such that $v(t_i(P))$ becomes arbitrary large as in 2.3. $\square$

(c) Since by assumption, all elements of A_G are in $\mathbf{Z}$, we can set $\widetilde{a} = a$, $\ell = n$ for any $a \in A_G$ to get the required norm-boundedness.

Remarks. (i) From available computer algebra, the construction of elliptic curves which fit the above can be automated. One can compute ranks of elliptic curves over $\mathbf{Q}$ quite fast using `mwrank` by J. Cremona [4], and over number fields using the `gp`-package of D. Simon [14]. One finds for example unconditionally that the curve $y^2 = x^3 + 8x$ has rank one over $\mathbf{Q}$ and this rank stays the same over $\mathbf{Q}(\sqrt[3]{2}), \mathbf{Q}(\sqrt[4]{2})$, hence the diophantine problem for the integers in these number fields has a negative answer (note that their Galois closures are non-abelian).

(ii) We ask: given K, can one construct in some clever way a curve C over $\mathbf{Q}$ such that its Jacobian satisfies the above conditions?

4.2 Rank-preservation over Z. A similar construction (of which we leave out the details) can be performed if there exists a commutative (not necessarily complete) group variety G over $\mathbf{Z}$ such that $G(\emptyset)$ is finitely generated and such that $\operatorname{rk} G(\mathbf{Z}) = \operatorname{rk} G(\emptyset) > 0$. As an example of this, let L be another number field, linearly disjoint from K. Let $\langle a_i \rangle$ denote a $\mathbf{Z}$-basis for $L/\mathbf{Q}$ (this is also a basis for $\emptyset_{KL}$ over $\emptyset_K$). Let T_L denote the norm one torus $N^L_{\mathbf{Q}}(\sum a_i x_i) = 1$. Then $T_L(\mathbf{Z}) \cong \emptyset_L^*$ and

$$T_L(\emptyset_K) = \ker(N_K^{KL} : \emptyset_{KL}^* \to \emptyset_K^*),$$

hence (by surjectivity of the relative norm) $\operatorname{rk} T_L(\emptyset_K) = \operatorname{rk} \emptyset_{KL}^* - \operatorname{rk} \emptyset_K^*$. In particular, $T_L(\emptyset_K) = T_L(\mathbf{Z})$ iff

$$r_{KL} + s_{KL} = r_K + s_K + r_L + s_L - 1$$

where r_M, s_M denote the number of real, respectively half the number of complex embeddings of a number field M.

(a) If K is totally real of degree d, $r_{KL} = dr_L, s_{KL} = ds_L$, hence we want $r_L + s_L = 1$, which we can achieve by choosing L quadratic imaginary; but then $T_L(\mathbf{Z})$ is of rank zero.

(b) If K is totally complex of degree d, $r_K = 0, s_K = \frac{d}{2}$. Also KL is then totally complex, so $r_{KL} = 0, s_{KL} = \frac{d}{2}[L : \mathbf{Q}]$, hence we want $\frac{d}{2}([L : \mathbf{Q}] - 1) = r_L + s_L - 1$, but since the right hand side is less than or equal to $[L : \mathbf{Q}] - 1$, we find $d \leq 2$. Hence this strategy only works for K quadratic imaginary.

The conclusion is that this approach covers Denef's result from [6], except that he can discard the first condition in our theorem (existence of elliptic curve of rank one) by using a torus instead.

Remark. In all these examples, division-ample sets are actually subsets of the integers. Can one find a division-ample set which does not consists of just ordinary integers?

Aknowledgements. The authors thank Jan Van Geel for very useful help and encouragement. The third author was supported by a Marie-Curie Individual Fellowship (HPMF-CT-2001-01384).

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